Asymptotic Completeness in Quantum Scattering Theory — Mourre Theory and Asymptotic Observables in Local Relativistic Quantum Field Theory

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The talk is based on the following papers:
Mourre theory and asymptotic observables in local rel. QFT. *Commun. Math. Phys.* 405 (2024)
Mourre theory and spectral analysis of energy-momentum operators in rel. QFT. *Lett. Math. Phys.* 114 (2024)

Overview

Primary Topic: Scattering Theory

Part 1: Quantum Mechanical Scattering Theory — *Scattering of non-relativistic many-body systems*

Part 2: Quantum Field Theory — *Algebraic quantum field theory, Haag–Ruelle scattering theory, and Araki–Haag detectors*

Part 3: Mourre's Conjugate Operator Method — *Positive commutator techniques*

Goal: To understand how a system of interacting particles evolves *asymptotically*.

Key concept: asymptotic completeness

Definition

Asymptotic completeness asserts that every state can be decomposed into bound and scattering states:

$$\mathcal{H} = \mathcal{H}_{\mathrm{bound}} \oplus \mathcal{H}_{\mathrm{scat}},$$

where $\mathcal{H}_{\mathrm{bound}}$ and $\mathcal{H}_{\mathrm{scat}}$ are the spaces of bound and scattering states, respectively.

Part 1

Quantum Mechanical Scattering Theory

Scattering of non-relativistic many-body systems

Quantum Mechanical Scattering Theory

Let H be a many-particle Hamiltonian (with centre-of-mass motion removed) on the Hilbert space $L^2(\mathbb{R}^{3(n-1)})$:

$$H = -\frac{1}{2}\Delta + \sum_a V_a(x^a),$$

where $-\Delta$ is the Laplacian, a is a cluster decomposition of $\{1, \ldots, n\}$, and V_a is a (many-body) interaction potential.

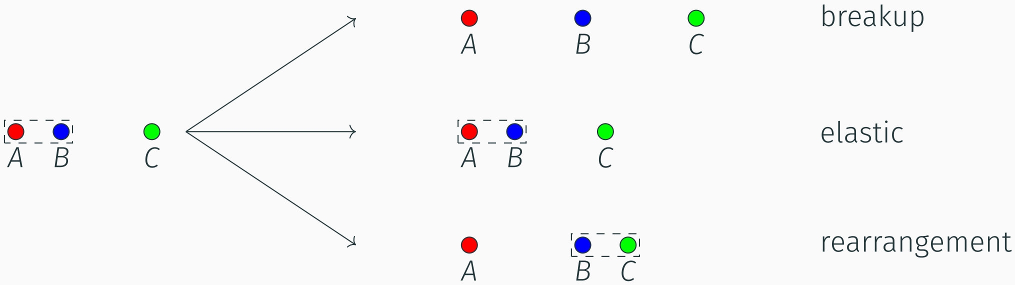
Asymptotic completeness is equivalent to **asymptotic clustering**:

$$\lim_{t \to \pm\infty} \| \mathrm{e}^{-\mathrm{i}tH} \psi - \sum_a \mathrm{e}^{-\mathrm{i}tH_a} \psi_{a,\pm} \| = 0,$$

where $H_a = -\Delta + \sum_{b \leq a} V_b(x^b)$ are cluster Hamiltonians.

In the two-particle case ($n = 2$), $\psi \in L^2(\mathbb{R}^3)$ is an eigenstate of H (bound state), or $\mathrm{e}^{-\mathrm{i}tH}\psi \sim \mathrm{e}^{-\mathrm{i}tH_0}\psi_\pm$ approaches a solution of the free system ($V = 0$) as $t \to \pm\infty$.

Three-Particle Scattering Channels



History of Quantum Mechanical Scattering Theory

1950s 2 particles (Kato, Rosenblum, Kuroda, …)

1959 *formulation of N-particle problem* (Hack)

1963 3 particles (Faddeev; Ginibre–Moulin ’74, Thomas ’75)

1978-84 3 particles, short- and long-range forces (Enss)

1981 *Mourre estimate* (Mourre; Perry–Sigal–Simon ’81, Froese–Herbst ’82)

1987 N particles, short-range (Sigal–Soffer; Graf ’90, Yafaev ’93)

1993 N particles, long-range (Dereziński; Sigal–Soffer ’94)

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Faddeev equations

$$R(z) = R_0(z) - R_0(z) \sum_{a,b} M_{ab} R_0(z),$$

$$M_{ab} = T_a \delta_{ab} + T_a R_0(z) \sum_{c \neq a} M_{cb},$$

$$T_a = V_a + V_a R_a(z) V_a.$$

1978-84 3 particles, sl…

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Phase-space analysis,
propagation estimates

$$\int_1^\infty \left\| \left(\frac{x_a}{t} - D_a \right) q_a \left(\frac{x}{t} \right) \psi_t \right\|^2 \frac{\mathrm{d}t}{t} \leq C \|\psi\|^2$$

Propagation of quantum particles is concentrated along classical trajectories.

1987 *N* particles, short-range (Sigal–Soffer; Graf '90, Yafaev '93)

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Mourre estimate

$$E(J)[H, \mathrm{i}A]E(J) \geq \theta E(J), \ \theta > 0,$$

E spectral measure of H.

1978-84 3 particles, sl

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Asymptotic Completeness in Quantum Field Theory

- ϕ^4_2 **model** (in two spacetime dimensions):
  - 2 particles [Spencer–Zirilli ’76]
  - 3 particles [Combescure–Dunlop ’82]
  - in finite volume [Dereziński–Gérard ’00]
- **Integrable models**
  - with factorising *S*-matrix [Lechner ’08]
- **Non-relativistic QFT**:
  - (confined) Pauli–Fierz Hamiltonian [Dereziński–Gérard ’97]
  - Rayleigh scattering [Fröhlich–Griesemer–Schlein ’02]
  - Compton scattering [Fröhlich–Griesemer–Schlein ’04]
  - Nelson model (below three-particle threshold) [Dybalski–Møller ’14]

Conceptual challenges:

Determining the particle content is already a difficult problem. It is not possible to read off the particle content from the Lagrangian or the equations of motions (e.g. solitons in ϕ_2^4).

QFT allows for processes that create or annihilate particles.

Technical challenges:

Dynamical properties of systems with non-quadratic dispersion relation are not well understood.

Asymptotic Completeness for Dispersive Systems

Open problem: Prove asymptotic completeness for dispersive Hamiltonians:

$$H = h(D) + \sum_a V_a(x^a).$$

Difficulty: Inter-cluster motion depends on the internal motion of the particles within the clusters. Separation of external and internal motion only in the quadratic case, $h(D) = D^2$.

Results:

- Asymptotic completeness in the two-particle case is proved.
- The Mourre estimate generalises [Dereziński '90, Gérard '91, Damak '97].
- Low- and large-velocity estimates can be established.

Part 2

Quantum Field Theory

Algebraic quantum field theory,
Haag–Ruelle scattering theory,
and Araki–Haag detectors

Algebraic Quantum Field Theory

The focus in algebraic quantum field theory is on observables.

Fundamental object: net of observables $\{\mathcal{A}(O)\}_O$, $\mathcal{A}(O) \subset \mathfrak{B}(\mathcal{H})$ von Neumann algebras, $O \subset \mathbb{R}^4$ spacetime regions.

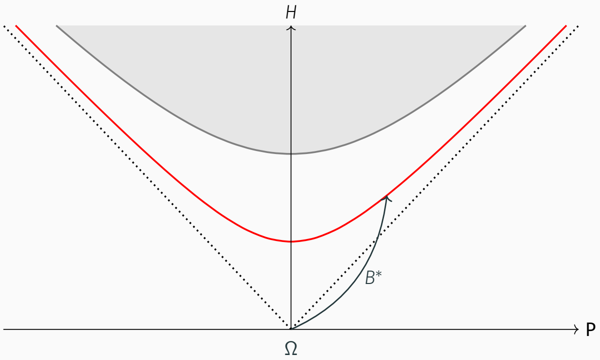
Properties of the net:

- Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: O_1, O_2 space-like separated $\Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$.
- Poincaré covariance: $\exists$ unitary rep. $U : \mathbb{R}^4 \rtimes \mathcal{L} \to \mathfrak{B}(H)$ s.t. $U(x,\Lambda)\mathcal{A}(O)U(x,\Lambda)^* = \mathcal{A}(\Lambda O + x)$.
- Vacuum vector: $\exists \Omega \in \mathcal{H} \backslash \{0\}$ s.t. $U(x,1)\Omega = \Omega$.
- Spectrum condition: $U(x,1) = \mathrm{e}^{-\mathrm{i}x\cdot P}$, the spectrum of the energy-momentum operators $P = (H, \mathbf{P})$ is contained in the forward light-cone.
    - Strong spectrum condition: There is an isolated mass shell of one-particle states in the energy-momentum spectrum.

Haag–Ruelle Scattering Theory — Creation Operators

Definition

Let $A \in \mathcal{A}(O)$ be a local observable. A **creation operator** B^* is an operator of the form $B^* = \int_{\mathbb{R}^4} f(x)A(x)\,\mathrm{d}x$, where $\mathrm{supp}(\hat{f})$ is close to the mass shell.



Haag–Ruelle Scattering Theory — Scattering States

Theorem

For all creation operators $B_1^*, \dots, B_n^*$, and all $f_1, \dots, f_n \in L^2(\mathbb{R}^3)$, the limits

$$\psi_1 \times \cdots \times \psi_n := \lim_{t\to\infty} B_{1,t}^*[f_{1,t}] \dots B_{n,t}^*[f_{n,t}]\Omega$$

exist, where $\psi_i = B_{i,0}[f_{i,0}]\Omega$, $f_{i,t}$ is a solution of the Klein–Gordon equation with initial data f_i, and $B_t^*[f_t] = \int_{\mathbb{R}^3} f_t(\mathsf{x}) B^*(t,\mathsf{x})\,\mathrm{d}\mathsf{x}$. The space of scattering states,

$$\mathcal{H}^{\text{out}} := \overline{\text{span}}\{\Omega, \psi_1 \times \cdots \times \psi_n \mid n \in \mathbb{N}, \psi_1, \dots, \psi_n \in \mathfrak{h}_m\},$$

is identical to the Fock space over the one-particle space $\mathfrak{h}_m$.

Asymptotic Completeness

Definition

A quantum field theory model is **asymptotically complete** if $\mathcal{H} = \mathcal{H}^{\mathrm{out}}$.

Establishing asymptotic completeness axiomatically is difficult because

- additional bound states embedded in the multi-particle spectrum may exist (e.g. solitons in ϕ_2^4),
- typically, quantum field theories have a rich superselection structure; pairs of charged particles may form states in the vacuum sector,
- pathological counterexamples (generalised free fields) fitting into the axiomatic setting exist.

Asymptotic Completeness

Model-independent strategy for proving asymptotic completeness:

1. **Identification of particle detectors**: identify observables that can be interpreted as particle detectors.
2. **Triggering by scattering states**: show that particle detectors can only be triggered by scattering states. Prove that every state in the orthogonal complement of the scattering states lies in the kernel of all particle detector.
3. **Accessibility of quantum states**: demonstrate that every quantum state can trigger at least one particle detector.

Detectors

Definition

A **detector** is an almost local observable C that annihilates the vacuum vector Ω (i.e. $C\Omega = 0$).

Definition

An observable A is **almost local** if there exists a sequence $(A_r)_{r\in\mathbb{N}}$ of observables, where $A_r \in \mathcal{A}(K_r)$ is localised in the double cone K_r of radius r, such that

$$\|A - A_r\| \leq C_N r^{-N}.$$

Typical example: $C = B^*B$, where $B^* = \int_{\mathbb{R}^4} f(x)A(x)\,\mathrm{d}x$ is a creation operator.

Araki–Haag Formula

Let $C = B^*B$ be a detector. The observable $C(t, \mathsf{x})$ converges weakly to 0 as $t \to \infty$ due to dispersion. We integrate over $\mathbb{R}^3$ to compensate for dispersion:

$$C(h; t) = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^3} h\left(\frac{\mathsf{x}}{t}\right) (B^*B)(\mathsf{x}) \, \mathrm{d}\mathsf{x} \, \mathrm{e}^{-\mathrm{i}tH}, \; h \in L^\infty(\mathbb{R}^3).$$

The integral is well-defined due to a uniform bound by Buchholz (1990).

Theorem (Araki–Haag formula, 1967)

Let $\phi, \psi \in \mathcal{H}^{\mathrm{out}}$ be scattering states. If $\omega(\mathsf{p}) = \sqrt{|\mathsf{p}|^2 + m^2}$, then

$$\lim_{t\to\infty} \langle \phi, C(h; t)\psi\rangle = (2\pi)^3 \int_{\mathbb{R}^3} h(\nabla\omega(\mathsf{p})) \langle \mathsf{p}|B^*B|\mathsf{p}\rangle \langle \phi, a^*_{\mathrm{out}}(\mathsf{p}) a_{\mathrm{out}}(\mathsf{p})\psi\rangle \, \mathrm{d}\mathsf{p}.$$

The r.h.s. is a particle counter with sensitivity $\langle \mathsf{p}|B^*B|\mathsf{p}\rangle$ and velocity filter h.

Convergence of Araki–Haag Detectors

Problem: Prove convergence of Araki–Haag detectors on arbitrary states.

Dybalski–Gérard (2014) obtained convergence of products of Araki–Haag detectors sensitive to particles with distinct velocities (i.e. $\mathrm{supp}(h_1) \cap \mathrm{supp}(h_2) = \emptyset$):

$$
\begin{aligned}
&C(h_1, t)C(h_2, t) \\
&\quad = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^3} h_1\left(\frac{\mathsf{x}}{t}\right)(B_1^* B_1)(\mathsf{x})\,\mathrm{d}\mathsf{x} \int_{\mathbb{R}^3} h_2\left(\frac{\mathsf{y}}{t}\right)(B_2^* B_2)(\mathsf{y})\,\mathrm{d}\mathsf{y}\,\mathrm{e}^{-\mathrm{i}tH}.
\end{aligned}
$$

They achieved this convergence result by translating **large-velocity** and **phase-space propagation estimates** from QM to QFT.

They did not cover the convergence of a single detector and detectors sensitive to particles with the same velocity due to a missing low-velocity propagation estimate.

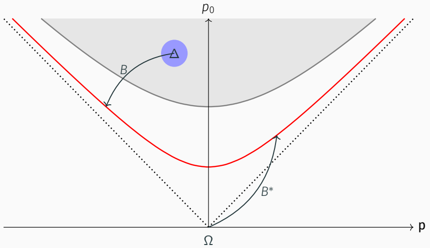
Convergence of Araki–Haag Detectors

Theorem (Kr24)

Let $\psi \in \mathcal{H}_{\mathrm{ac}}(H, \mathsf{P})$ have bounded energy. If $B\psi$ is a one-particle state, then

$$C(h,t)\psi = \mathrm{e}^{\mathrm{i}tH} \int_{\mathbb{R}^3} h\left(\frac{\mathbf{x}}{t}\right) (B^*B)(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{e}^{-\mathrm{i}tH} \psi$$

converges as $t \to \infty$. The limit is 0 if $\psi \in (\mathcal{H}^{\mathrm{out}})^{\perp}$.



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Spectral decomposition of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}}(H, \mathsf{P}) \oplus \mathcal{H}_{\mathrm{ac}}(H, \mathsf{P}) \oplus \mathcal{H}_{\mathrm{sc}}(H, \mathsf{P}).$$

Typically, $\mathcal{H}_{\mathrm{pp}}(H, \mathsf{P}) = \mathrm{span}\{\Omega\}$, $\mathcal{H}_{\mathrm{sc}}(H, \mathsf{P}) \leftrightarrow$ mass shells

$\mathcal{H}_{\mathrm{sc}}(H, \mathsf{P})$ may contain exotic states for which we cannot control convergence.

Main Steps of the Proof — Insertion of a Second Detector

1. The proof of the theorem reduces to the L^2-convergence of

$$e^{-\,\mathrm{i}t(\omega(D_{\mathsf{x}})+\omega(D_{\mathsf{y}}))}\langle e^{-\,\mathrm{i}tH}\psi, B^*(\mathsf{x})B_2^*(\mathsf{y})\Omega\rangle,\ \omega(D_{\mathsf{x}}) = \sqrt{|D_{\mathsf{x}}|^2+m^2}.$$

(Similar to existence/completeness of wave operators in QM.)

Lemma

If $\Delta \subset \mathbb{R}^4$ is a compact set sufficiently close to the mass shell, then there is a creation operator B_2^* such that

$$E(\Delta) = E(\Delta)\int_{\mathbb{R}^3}(B_2^*B_2)(\mathsf{y})\,\mathrm{d}\mathsf{y}\,E(\Delta),$$

where E is the spectral measure of the energy-momentum operators P.

Main Steps of the Proof — Cook's Method

2. Prove the Cauchy property by **Cook's method** (i.e. prove that the t-derivative is integrable) and formulate expression in relative coordinates (u, v):

$$\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}D_{\mathsf{v}}+D_{\mathsf{u}})+\omega(\frac{1}{2}D_{\mathsf{v}}-D_{\mathsf{u}}))} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H}\, \mathrm{e}^{-\mathrm{i}\mathsf{v}\cdot\mathsf{P}} \phi(\mathsf{u}) \rangle \,\mathrm{d}\tau,$$

$$\phi(\mathsf{u}) \sim [B^*(\mathsf{u}), B_2^*]\Omega.$$

The commutator $[B^*(\mathsf{u}), B_2^*]\Omega$ arises from the fact that

$$HB_2^*(\mathsf{y})\Omega = \omega(D_{\mathsf{y}})B_2^*(\mathsf{y})\Omega$$

because $B_2^*(\mathsf{y})\Omega$ is a one-particle state. Observe that, due to locality,

$$\|[B^*(\mathsf{u}), B_2^*]\| \leq C_N \langle \mathsf{u} \rangle^{-N}.$$

3. Remove the centre-of-mass motion through fibration over total momentum D_{v} (i.e. take the Fourier transformation $\mathcal{F}_{\mathsf{v}\to\mathsf{p}}$):

$$\omega_{\mathsf{p}}(D_{\mathsf{u}}) = \omega(\mathsf{p}/2 + D_{\mathsf{u}}) + \omega(\mathsf{p}/2 - D_{\mathsf{u}}).$$

We must prove convergence of the following expression:

$$\int_{\mathbb{R}^s}\int_{\mathbb{R}^s}\left|\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{i}\tau(\omega(\frac{1}{2}\mathsf{p}+D_{\mathsf{u}})+\omega(\frac{1}{2}\mathsf{p}-D_{\mathsf{u}}))}\mathcal{F}_{\mathsf{v}\to\mathsf{p}}\langle\psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathsf{v}\cdot\mathsf{P}}\phi(\mathsf{u})\rangle\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathsf{u}\,\mathrm{d}\mathsf{p}$$
$$= \int_{\mathbb{R}^s}\sup_{\|f\|_{L^2}=1}\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^s}\overline{f(\mathsf{u})}\,\mathrm{e}^{-\mathrm{i}\tau\omega_{\mathsf{p}}(D_{\mathsf{u}})}\mathcal{F}_{\mathsf{v}\to\mathsf{p}}\langle\psi, \mathrm{e}^{\mathrm{i}\tau H}\,\mathrm{e}^{-\mathrm{i}\mathsf{v}\cdot\mathsf{P}}\phi(\mathsf{u})\rangle\,\mathrm{d}\mathsf{u}\,\mathrm{d}\tau\right|^2 \mathrm{d}\mathsf{p}.$$

Main Steps of the Proof — Local Decay Estimate

4. Apply the Cauchy–Schwarz inequality ($\nu > 1/2$):

$$\int \left(\sup_{\|f\|_{L^2}=1} \int_{t_1}^{t_2} \|(1+|A_\mathsf{p}|)^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_\mathsf{p}(D_\mathsf{u})} f\|_{L^2}^2 \, \mathrm{d}\tau \right) \times \left(\int_{t_1}^{t_2} \|(1+|A_\mathsf{p}|)^{\nu} \mathcal{F}_{\mathsf{v}\to\mathsf{p}} \langle \psi, \mathrm{e}^{\mathrm{i}\tau H} \, \mathrm{e}^{-\mathrm{i}\mathsf{v}\cdot\mathsf{P}} \phi \rangle \|_{L^2}^2 \, \mathrm{d}\tau \right) \mathrm{d}\mathbf{p},$$

where $A_\mathsf{p} \sim \mathbf{u}$ is a modified generator of dilations. The second factor is controlled by **locality** (QFT), the first by a **local decay estimate** (Mourre's method, QM):

$$\int_{-\infty}^{\infty} \|(1+|A_\mathsf{p}|)^{-\nu} \, \mathrm{e}^{\mathrm{i}\tau\omega_\mathsf{p}(D_\mathsf{u})} f\|_{L^2}^2 \, \mathrm{d}\tau \le C \|f\|_{L^2}^2.$$

Part 3

Mourre's Conjugate Operator Method

Positive commutator techniques

Mourre's conjugate operator method is a powerful mathematical technique from spectral theory, which is based on a strictly positive commutator estimate.

Definition

Let H, A be self-adjoint operators. H obeys a **Mourre estimate** with **conjugate operator** A on an open set $J \subset \mathbb{R}$ if

$$E(J)[H, \mathrm{i}A]E(J) \geq aE(J), \ \ a > 0,$$

where E is the spectral measure of H.

Typical example from QM: $H = -\Delta$, $A = (x \cdot D + D \cdot x)/2 \Rightarrow [-\Delta, \mathrm{i}A] = -2\Delta$.

Key Results from the Mourre Estimate

Assume H obeys a Mourre estimate on J with conjugate operator A. Let $K \subset J$ be compact and $\nu > 1/2$. Key results from the Mourre estimate:

- **Limiting absorption principle (LAP):**

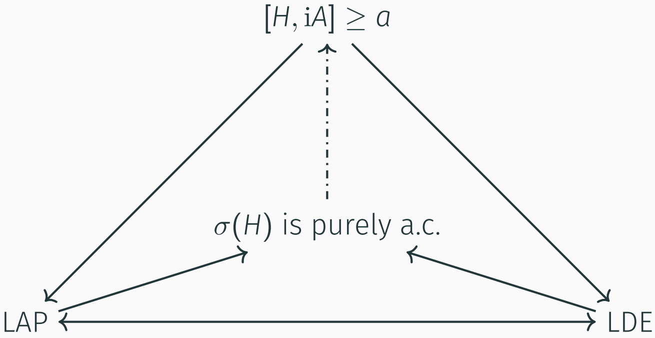
$$\sup_{\lambda \in K, \mu > 0} \|(1+|A|)^{-\nu}(H - \lambda \mp \mathrm{i}\mu)^{-1}(1+|A|)^{-\nu}\| < \infty.$$

- **Local decay estimates (LDE):**

$$\int_{\mathbb{R}} \|(1+|A|)^{-\nu}\, \mathrm{e}^{\mathrm{i}tH} E(K) f\|^2 \,\mathrm{d}t \leq C_K \|f\|^2.$$

- **Absolutely continuous spectrum:** The spectrum of H in J is purely abs. cont.

Triangle of Mourre Theory



Mourre's method can be applied to analyse spectral properties of the energy-momentum operators (H, P) in quantum field theory.

Commutation relation of momentum P_j and Lorentz boost K_j in direction j:

$$[P_j, \mathrm{i}K_j] = H \geq 0.$$

Mourre Estimate:

$$E(|P_j| \geq a)[P_j, \mathrm{i}K_j]E(|P_j| \geq a) \geq aE(|P_j| \geq a), \quad a > 0$$

Conclusion: The spectrum of P_j outside $\{0\}$ is purely absolutely continuous.

Applications in Quantum Field Theory

Further results:

- It is possible to apply Mourre's method to prove that the joint spectrum of P is purely absolutely continuous away from 0.
- There is also a Mourre estimate for the Hamiltonian H in quantum field theory.
    - The conjugate operator is more complicated in this case. It is constructed from the energy-momentum operators and the Lorentz boosts.
    - This is an interesting result because the Hamiltonian in quantum field theory is typically rather abstract. The Hamiltonian is obtained through a limiting procedure from renormalisation or defined axiomatically as the generator of time translations.
    - A Mourre estimate had previously been proved by Dereziński–Gérard (2000) for the Hamiltonian of the ϕ_2^4 model in finite volume.

Summary

Conclusion and Outlook

- **Asymptotic completeness** is a challenging open problem in local relativistic quantum field theory (QFT).
- There is a strategy to establish asymptotic completeness that uses **particle detectors** (Araki–Haag detectors).
- We proved the **convergence of a single Araki–Haag detector** for arbitrary states $\psi \in \mathcal{H}_{\mathrm{ac}}(H, \mathsf{P})$ below the three-particle threshold. This result is a key prerequisite for proving asymptotic completeness.
- Mourre's method can be utilised in spectral and scattering theory of QFT.
- Possibly, further methods from many-body quantum mechanics can be transferred to QFT to prove the convergence of Araki–Haag detectors in the multi-particle region.